On the reverse mathematics of Peano categoricity

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Outline

1 Introduction
   - Peano system
   - Reverse Mathematics

2 RM of Peano categoricity
   - over RCA_0
   - over RCA_0^*

3 Other categoricity theorems
   - system of order
   - system of order and successor function
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2. RM of Peano categoricity
   - over RCA
   - over RCA*

3. Other categoricity theorems
   - system of order
   - system of order and successor function
Peano system is a system of a successor function and second-order induction axiom.

**Definition**

A Peano system is a triple \((M, e, f)\) where \(M\) is a set, \(e \in M\) and \(f : M \rightarrow M\) is a function such that

1. \(f\) is one-to-one,
2. \(\forall x \in M \ f(x) \neq e\),
3. induction: for any \(Z \subseteq M\),

\[ (e \in Z \land \forall x \in Z(f(x) \in Z)) \rightarrow \forall x(x \in Z). \]
Dedekind’s theorem

It is well-known that Peano system is categorical in the following sense.

**Theorem (Dedekind 1888)**

*Any two Peano system is isomorphic.*

*In other words, every Peano system is isomorphic to $\mathbb{N} = (\mathbb{N}, 0, S)$.*

**Proof.**

Let $(M, e, f)$ be a Peano system.
Define $A = \{x \in M \mid \exists n \in \mathbb{N} \ f^n(e) = x\}$.
Then, by induction, $A = M$.
$\Phi(x) = \min\{n \mid f^n(e) = x\}$ is the desired isomorphism.
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What is needed for second-order characterization?

The Peano categoricity theorem means that "Peano system characterizes the natural number system."

In other words, the natural number system is second-order characterizable by using a successor function.

From the viewpoint of second-order characterization, Väänänen raised the following question.

**Question (Jouko Väänänen)**

Which axioms are needed to prove the Peano categoricity theorem in second-order arithmetic? (This should be weak.)

See, "Second order logic or set theory?" by Jouko Väänänen, to appear in BSL.

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Subsystems of second-order arithmetic

Language of second order arithmetic ($\mathcal{L}_2$):

- number variables: $x, y, z, \ldots$
- set variables: $X, Y, Z, \ldots$
- constants and functions: $0, 1, +, \cdot$
- relations: $=, <, \in$

Review (Big five plus one)

- **RCA$_0$**: basic axioms: “discrete ordered semi-ring”
  + $\Sigma^0_1$ induction + recursive comprehension.
- **WWKL$_0$**: RCA$_0$ + weak weak König’s lemma.
- **WKL$_0$**: RCA$_0$ + weak König’s lemma.
- **ACA$_0$**: RCA$_0$ + arithmetical comprehension.
- **ATR$_0$**: RCA$_0$ + arithmetical transfinite recursion.
- **$\Pi^1_1$CA$_0$**: RCA$_0$ + $\Pi^1_1$-comprehension.
Theorem

The following are provable within RCA₀.

1. The structure theorem for finitely generated abelian group.
2. Mean value theorem.
3. Implicit function theorem.
4. Taylor’s expansion theorem for holomorphic function.
5. The Riemann mapping theorem for a polygonal region.
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Reverse Mathematics

Theorem

The following are equivalent over RCA₀.

1. WKL₀.
2. Completeness theorem/ compactness theorem.
3. Uniqueness of algebraic closures of a countable field.
5. The Cauchy integral theorem for a Jordan curve.
6. The Riemann mapping theorem for a Jordan region.
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The following are equivalent over RCA₀.

1. ACA₀.
2. Ramsey’s theorem: RTₙ for n ∈ ω.
3. Every countable commutative ring has a maximal ideal.
4. Every normal family 𝐹_𝐃, i.e., 𝐹 is a family of uniformly bounded holomorphic functions on a common domain D ⊆ ℂ, has a uniformly convergent sub sequence.
5. The Riemann mapping theorem (over WKL₀).
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Theorem

The following are equivalent over RCA₀.

1. ACA₀.
2. Ramsey’s theorem: RTₙ for n ∈ ω.
3. Every countable commutative ring has a maximal ideal.
4. Every normal family Fₗ, i.e., F is a family of uniformly bounded holomorphic functions on a common domain D ⊆ ℂ, has a uniformly convergent subsequence.
5. The Riemann mapping theorem (over WKL₀).
6. . .
Theorem (Harrington)

Either $\text{RCA}_0$ or $\text{WKL}_0$ is a $\Pi^1_1$-conservative extension of $\text{I} \Sigma_1$.

Theorem (Friedman)

Either $\text{RCA}_0$ or $\text{WKL}_0$ is a $\Pi^0_2$-conservative extension of Primitive Recursive Arithmetic (PRA). Thus, they are proof-theoretically equivalent to PRA.

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$\text{ACA}_0$ is a $\Pi^1_1$-conservative extension of $\text{PA}$. 
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1. \(f\) is one-to-one,
2. \(\forall x \in M \ f(x) \neq e,\)
3. induction: for any \(Z \subseteq M,
   \[ (e \in Z \land \forall x \in Z (f(x) \in Z)) \to \forall x (x \in Z). \]

Note that \(\text{RCA}_0\) proves \(\mathbb{N} = (\mathbb{N}, 0, S)\) is a Peano system.

A Peano system \((M, e, f)\) is said to be isomorphic to \(\mathbb{N}\) if there exists a bijective function \(\Phi : M \to \mathbb{N}\) such that \(\Phi(e) = 0\) and \(\Phi(f(x)) = \Phi(x) + 1\).
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What is needed for PCT?

PCT: every Peano system is isomorphic to \( \mathbb{N} \).

Observation

ACA\(_0\) proves PCT.

Proof.

Let \((M, e, f)\) be a Peano system.
Define \( A = \{ x \in M \mid \exists n \in \mathbb{N} \ f^n(e) = x \} \).
Then, by induction, \( A = M \).
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Then, does the converse hold?
\( \Rightarrow \) No!
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Definition (RCA₀)

A Peano system \((M, e, f)\) is said to be almost isomorphic to \(\mathbb{N}\) if for any \(x \in M\) there exists a sequence \(\langle a_i \mid i \leq k \rangle\) such that \(a_0 = e\), \(a_k = x\) and \(a_{i+1} = f(a_i)\) for any \(i < k\).

wPCT: every Peano system is almost isomorphic to \(\mathbb{N}\).
ISO: every Peano system which is almost isomorphic to \(\mathbb{N}\) is isomorphic to \(\mathbb{N}\).

Then,

\[\text{PCT} = \text{wPCT} + \text{ISO}.\]
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A Peano system $(M, e, f)$ is said to be almost isomorphic to $\mathbb{N}$ if for any $x \in M$ there exists a sequence $\langle a_i \mid i \leq k \rangle$ such that $a_0 = e$, $a_k = x$ and $a_{i+1} = f(a_i)$ for any $i < k$.

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Lemma

\( \text{RCA}_0 \) proves ISO.

Proof.

Let \((M, e, f)\) be a Peano system which is almost isomorphic to \(\mathbb{N}\). Then, \(M = \{x \in M \mid \exists n \in \mathbb{N} \ f^n(e) = x\}\).

Thus, \(\Phi(x) = \min\{n \mid f^n(e) = x\}\) is the desired isomorphism.
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What is needed for PCT?

**Theorem**

\[ \text{WKL}_0 \text{ proves } \text{wPCT}. \]

**Proof.**

Assume \((M, e, f)\) is not almost isomorphic to \(\mathbb{N}\). Then, there exists \(c \in M\) such that \(f^n(e) \neq c\) for any \(n \in \mathbb{N}\). Define a tree \(T \subseteq 2^{<\mathbb{N}}\) as follows:

\[
T = \{ \sigma \mid e < |\sigma| \rightarrow \sigma(e) = 1, \ c < |\sigma| \rightarrow \sigma(c) = 0, \\
(x, y \in M \land x, y < |\sigma| \land f(x) = y) \rightarrow \sigma(x) = \sigma(y) \}
\]

Then, \(T\) is infinite, thus, \(T\) has a path \(h\).

Let \(A = \{ x \in M \mid h(x) = 1 \}\). Then, \(e \in A\) and \(A\) is closed under \(f\), but \(c \notin A\). Thus, \((M, e, f)\) is not a Peano system. \(\square\)
What is needed for PCT?

**Theorem**

$\text{WKL}_0$ proves $wPCT$.

**Proof.**

Assume $(M, e, f)$ is not almost isomorphic to $\mathbb{N}$. Then, there exists $c \in M$ such that $f^n(e) \neq c$ for any $n \in \mathbb{N}$.

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**Theorem**

\( \text{WKL}_0 \) proves \( w\text{PCT} \).

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T = \{ \sigma \mid e < |\sigma| \rightarrow \sigma(e) = 1, \ c < |\sigma| \rightarrow \sigma(c) = 0, \ (x, y \in M \land x, y < |\sigma| \land f(x) = y) \rightarrow \sigma(x) = \sigma(y) \}
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Then, $T$ is infinite, thus, $T$ has a path $h$.
Let $A = \{ x \in M \mid h(x) = 1 \}$. Then, $e \in A$ and $A$ is closed under $f$, but $c \notin A$. Thus, $(M, e, f)$ is not a Peano system.
What is needed for PCT?

**Theorem**

\( WKL_0 \) proves \( \text{wPCT} \).

**Proof.**

Assume \((M, e, f)\) is not almost isomorphic to \( \mathbb{N} \). Then, there exists \( c \in M \) such that \( f^n(e) \neq c \) for any \( n \in \mathbb{N} \). Define a tree \( T \subseteq 2^{<\mathbb{N}} \) as follows:

\[
T = \{ \sigma \mid e < |\sigma| \rightarrow \sigma(e) = 1, \ c < |\sigma| \rightarrow \sigma(c) = 0, \ (x, y \in M \land x, y < |\sigma| \land f(x) = y) \rightarrow \sigma(x) = \sigma(y) \}
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Then, \( T \) is infinite, thus, \( T \) has a path \( h \).

Let \( A = \{ x \in M \mid h(x) = 1 \} \). Then, \( e \in A \) and \( A \) is closed under \( f \), but \( c \notin A \). Thus, \((M, e, f)\) is not a Peano system. \( \square \)
What is needed for PCT?

**Theorem**

WKL$_0$ proves $wPCT$.

**Proof.**

Assume $(M, e, f)$ is not almost isomorphic to $\mathbb{N}$. Then, there exists $c \in M$ such that $f^n(e) \neq c$ for any $n \in \mathbb{N}$. Define a tree $T \subseteq 2^{<\mathbb{N}}$ as follows:

$$T = \{ \sigma \mid e < |\sigma| \rightarrow \sigma(e) = 1, \; c < |\sigma| \rightarrow \sigma(c) = 0, \; (x, y \in M \land x, y < |\sigma| \land f(x) = y) \rightarrow \sigma(x) = \sigma(y) \}$$

Then, $T$ is infinite, thus, $T$ has a path $h$.

Let $A = \{ x \in M \mid h(x) = 1 \}$. Then, $e \in A$ and $A$ is closed under $f$, but $c \notin A$. Thus, $(M, e, f)$ is not a Peano system. 

\qed
What is needed for PCT?

**Theorem**

*Over RCA₀, wPCT implies WKL₀.*

**Proof.**

We show \( \neg \text{WKL} \rightarrow \neg \text{wPCT}. \)

Let \( T \subseteq 2^{\mathbb{N}} \) be an infinite tree which has no infinite path.

Then, there exists \( n \in \mathbb{N} \) such that \( 1^n \notin T \).

Let \( \tau_0 \) be the shortest such string, and let \( \bar{T} = T \cup \{1^n \mid n \in \mathbb{N}\} \).

Consider the lexicographic order \( <_{lx} \) on \( \bar{T} \), and let \( f : \bar{T} \rightarrow \bar{T} \) be a successor function with respect to this order.

Note that \( f \) can be defined by \( \Delta^0_1\)-CA, since \( f(\sigma) \) is one of \( \{\sigma \uparrow 0, \sigma \uparrow 1\} \cup \{\sigma \uparrow i \uparrow 1 \mid i < |\sigma|\} \).

Then, \( (\bar{T}, \langle \rangle, f) \) is a Peano system by the following claim, and it is not almost isomorphic to \( \mathbb{N} \) since \( \forall n \in \mathbb{N} \ f^n(\langle \rangle) \neq \tau_0 \).
What is needed for PCT?

Theorem

Over \( \text{RCA}_0 \), \( \text{wPCT} \) implies \( \text{WKL}_0 \).

Proof.

We show \( \neg \text{WKL} \rightarrow \neg \text{wPCT} \).

Let \( T \subseteq 2^{<\mathbb{N}} \) be an infinite tree which has no infinite path. Then, there exists \( n \in \mathbb{N} \) such that \( 1^n \notin T \).

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Consider the lexicographic order \( <_{lx} \) on \( \tilde{T} \), and let \( f : \tilde{T} \rightarrow \tilde{T} \) be a successor function with respect to this order.

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Then, \( (\tilde{T}, \langle \rangle, f) \) is a Peano system by the following claim, and it is not almost isomorphic to \( \mathbb{N} \) since \( \forall n \in \mathbb{N} \ f^n(\langle \rangle) \neq \tau_0 \).
What is needed for PCT?

**Theorem**

*Over RCA₀, wPCT implies WKL₀.*

**Proof.**

We show \( \neg WKL \rightarrow \neg wPCT. \)

Let \( T \subseteq 2^{<\mathbb{N}} \) be an infinite tree which has no infinite path.

Then, there exists \( n \in \mathbb{N} \) such that \( 1^n \notin T \).

Let \( \tau_0 \) be the shortest such string, and let \( \bar{T} = T \cup \{1^n \mid n \in \mathbb{N}\}. \)

Consider the lexicographic order \( <_{lx} \) on \( \bar{T} \), and let \( f : \bar{T} \rightarrow \bar{T} \) be a successor function with respect to this order.

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What is needed for PCT?

**Theorem**

*Over RCA*$_0$, wPCT implies WKL$_0$.

**Proof.**

We show $\neg$WKL $\rightarrow \neg$wPCT.
Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree which has no infinite path. Then, there exists $n \in \mathbb{N}$ such that $1^n \notin T$.
Let $\tau_0$ be the shortest such string, and let $\bar{T} = T \cup \{1^n \mid n \in \mathbb{N}\}$.
Consider the lexicographic order $<_{lx}$ on $\bar{T}$, and let $f : \bar{T} \rightarrow \bar{T}$ be a successor function with respect to this order.
Note that $f$ can be defined by $\Delta^0_1$-CA, since $f(\sigma)$ is one of \{\sigma \uparrow 0, \sigma \uparrow 1\} $\cup$ \{\sigma \uparrow i \uparrow 1 \mid i < |\sigma|\}.

Then, $(\bar{T}, \langle \rangle, f)$ is a Peano system by the following claim, and it is not almost isomorphic to $\mathbb{N}$ since $\forall n \in \mathbb{N} f^n(\langle \rangle) \neq \tau_0$. 
What is needed for PCT?

**Theorem**

*Over RCA₀, wPCT implies WKL₀.*

**Proof.**

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Consider the lexicographic order $<_\text{lx}$ on $\bar{T}$, and let $f : \bar{T} \rightarrow \bar{T}$ be a successor function with respect to this order.

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What is needed for PCT?

**Theorem**

*Over RCA*$_0$, wPCT implies WKL$_0$.

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Then, \( (\bar{T}, \langle \rangle, f) \) is a Peano system by the following claim,
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What is needed for PCT?

**Theorem**

Over $\text{RCA}_0$, $\text{wPCT}$ implies $\text{WKL}_0$.  

**Proof.**

We show $\neg \text{WKL} \rightarrow \neg \text{wPCT}$.  
Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree which has no infinite path. Then, there exists $n \in \mathbb{N}$ such that $1^n \not\in T$.  
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Consider the lexicographic order $<_{lx}$ on $\bar{T}$, and let $f : \bar{T} \rightarrow \bar{T}$ be a successor function with respect to this order.  
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What is needed for PCT?

**Proof (continued).**

**Claim:** there is no $A \subseteq \tilde{T}$ such that $\langle \rangle \in A$ and $A$ is closed under $f$.

If such $A$ exists, take $\tau \in \tilde{T} \setminus A$, and let $\tilde{A} = \{ \sigma \in A \mid \sigma \prec_{lx} \tau \}$. Then, $\tilde{A}$ is closed under $f$, and $\tilde{A} \subseteq T$.

Define $h : \mathbb{N} \to 2^{<\mathbb{N}}$ as $h(0) = \langle \rangle$ and

$$h(n + 1) = \begin{cases} h(n)^1 & \text{if } h(n)^1 \in \tilde{A}, \\ h(n)^0 & \text{otherwise.} \end{cases}$$

Then, $h(n) \in \tilde{A} \subseteq T$ for any $n \in \mathbb{N}$, since if $h(n)^1 \notin \tilde{A}$ then $h(n)^0 = f(h(n)) \in \tilde{A}$, but we assumed that $T$ has no path. □

**Corollary.**

PCT is equivalent to WKL$_0$ over RCA$_0$. 
Proof (continued).

**Claim:** there is no $A \subseteq \tilde{T}$ such that $\langle \rangle \in A$ and $A$ is closed under $f$.

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**Corollary**

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**Corollary**

PCT is equivalent to WKL\(_0\) over RCA\(_0\).
What is needed for PCT?

Proof (continued).

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Corollary

PCT is equivalent to WKL$_0$ over RCA$_0$.
What is needed for PCT?

Proof (continued).

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**Corollary**

PCT is equivalent to $\text{WKL}_0$ over $\text{RCA}_0$. ⌣
What is needed for PCT?

Proof (continued).

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Corollary

PCT is equivalent to WKL$_0$ over RCA$_0$. 

□
The strength of PCT

**Review**

\[ \text{WKL}_0 \text{ is a } \Pi^0_2\text{-conservative extension of Primitive Recursive Arithmetic (PRA), thus, it is proof-theoretically equivalent to PRA.} \]

Thus, we can say the following:

“We can characterize/redefine the natural number system by a successor function in a weak standpoint.”

**Question**

Is \( \text{WKL}_0 \) exactly the weakest in the sense of proof-theoretic strength?

Yes, in the following sense.
WKL₀ is a $\Pi^0_2$-conservative extension of Primitive Recursive Arithmetic (PRA), thus, it is proof-theoretically equivalent to PRA.

Thus, we can say the following:

“We can characterize/redefine the natural number system by a successor function in a weak standpoint.”

Is WKL₀ exactly the weakest in the sense of proof-theoretic strength?

Yes, in the following sense.
The strength of PCT

**Review**

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Thus, we can say the following:

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**Question**

Is $WKL_0$ exactly the weakest in the sense of proof-theoretic strength?

Yes, in the following sense.
The strength of PCT

Review

\( \text{WKL}_0 \) is a \( \Pi^0_2 \)-conservative extension of Primitive Recursive Arithmetic (PRA), thus, it is proof-theoretically equivalent to PRA.

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Question

Is \( \text{WKL}_0 \) exactly the weakest in the sense of proof-theoretic strength?

Yes, in the following sense.
We weaken the base system.

### Review (Big five)

<table>
<thead>
<tr>
<th>System</th>
<th>Description</th>
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<tr>
<td>$\text{RCA}_0$</td>
<td>basic axioms: “discrete ordered semi-ring” + $\Sigma^0_1$ induction + recursive comprehension.</td>
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<tr>
<td>$\text{WKL}_0$</td>
<td>$\text{RCA}_0$ + weak König’s lemma.</td>
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<td>$\Pi^1_1\text{CA}_0$</td>
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RM in a weaker base system

We weaken the base system.

**Definition**

- **RCA\(^*\)_0**: basic axioms: “discrete ordered semi-ring”
  + “for any \(x\), \(2^x\) exists” + \(\Sigma^0_0\)-induction
  + recursive comprehension.
- **RCA\(_0\)**: RCA\(^*\)_0 + \(\Sigma^0_1\)-induction.
- **WKL\(^*\)_0**: RCA\(^*\)_0 + weak König’s lemma.
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- **ATR\(_0\)**: RCA\(^*\)_0 + arithmetical transfinite recursion.
- **\(\Pi^1_1\)CA\(_0\)**: RCA\(^*\)_0 + \(\Pi^1_1\)-comprehension.
Theorem (Simpson-Smith)

The following are equivalent over $\text{RCA}_0^*$.

1. $\text{RCA}_0$.
2. Bounded $\Sigma^0_1$-comprehension.
3. For every countable field $K$, every polynomial $f(x) \in K[x]$ has only finitely many roots in $K$.
4. Every finitely generated vector space over a countable field has a basis.
5. Every finitely generated torsion-free abelian group is of the form $\mathbb{Z}^n$. 
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Theorem (Nemoto)

*The following are equivalent over RCA*$_0^*$.

1. WKL$_0^*$.
2. $\Sigma^0_1$-determinacy in Cantor space.
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Either $\text{RCA}_0^*$ or $\text{WKL}_0^*$ is a $\Pi^1_1$-conservative extension of $\text{BS}_{1} + \exp$.

Theorem (Simpson-Smith)

Either $\text{RCA}_0^*$ or $\text{WKL}_0^*$ is a $\Pi^0_2$-conservative extension of Elementary Function Arithmetic (EFA).

Thus, they are proof-theoretically equivalent to EFA, which is weaker than PRA.
Theorem (Simpson-Smith)

Either $\text{RCA}_0^*$ or $\text{WKL}_0^*$ is a $\Pi^1_1$-conservative extension of $\text{B} \Sigma_1 + \text{exp}$.

Theorem (Simpson-Smith)

Either $\text{RCA}_0^*$ or $\text{WKL}_0^*$ is a $\Pi^0_2$-conservative extension of Elementary Function Arithmetic (EFA). Thus, they are proof-theoretically equivalent to EFA, which is weaker than PRA.
What is needed for PCT?

We have already proved that $wPCT$ is equivalent to $WKL_0$ over $RCA_0$.

In fact, we can prove the following.

**Theorem**

$wPCT$ is equivalent to $WKL_0^*$ over $RCA_0^*$.

On the other hand, we can refine the following lemma.

**Lemma (review)**

$RCA_0$ proves $ISO$.

**Theorem**

$ISO$ is equivalent to $RCA_0$ over $RCA_0^*$. 
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\( \text{RCA}_0 \) proves \( \text{ISO} \).

**Theorem**

\( \text{ISO} \) is equivalent to \( \text{RCA}_0 \) over \( \text{RCA}^*_0 \).
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We have already proved that \( wPCT \) is equivalent to \( WKL_0 \) over \( \text{RCA}_0 \).
In fact, we can prove the following.

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What is needed for PCT?

Proof.

We have already proved $\text{RCA}_0 \rightarrow \text{ISO}$.

To show $\text{ISO} \rightarrow \text{RCA}_0$, we only need to show that for any infinite subset $A \subseteq \mathbb{N}$, there exists a one-to-one function $h : \mathbb{N} \rightarrow A$. (This is equivalent to $\Sigma^0_1$-induction.)

For given infinite $A \subseteq \mathbb{N}$, define $e = \min A$ and $f(x) = \min\{y \in A \mid y > x\}$.

Then, $(A, e, f)$ is a Peano system which is almost isomorphic to $\mathbb{N}$. Thus, isomorphism $\Phi^{-1} : \mathbb{N} \rightarrow A$ is a one-to-one function. □
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The strength of PCT

Corollary

PCT is equivalent to WKL₀ over RCA₀*. In fact, PCT splits as follows.

\[
\begin{align*}
\text{PCT} & = \text{ISO} + \text{wPCT} \\
\vdots & \quad \vdots \\
\text{WKL₀} & = \text{RCA₀} + \text{WKL₀}^*.
\end{align*}
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Thus, we can say the following:

“To characterize/redefine the natural number system by a successor function, the primitive recursion/ \( \Sigma^0_1 \)-induction is essentially needed.”
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\textbf{PCT is equivalent to WKL}_0 \textit{ over } RCA^*_0.\textbf{.}

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Outline

1. Introduction
   - Peano system
   - Reverse Mathematics

2. RM of Peano categoricity
   - over RCA₀
   - over RCA₀^*

3. Other categoricity theorems
   - system of order
   - system of order and successor function
Inductive ordered system

We can characterize the natural number system by using a linear order.

Definition (RCA₀*)

An ordered system is a triple \((M, e, <)\) where \(M \subseteq \mathbb{N}\), \(e \in M\) and \(< \subseteq M \times M\) is a relation such that

1. \(<\) is a linear order, and \(e\) is the minimum element,
2. for any \(x \in M\), the successor \(x' := \min\{y \in M \mid x < y\}\) exists,

Note that the successor function \(f(x) = x'\) may not exist.
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Note that the successor function \(f(x) = x'\) may not exist.
Inductive ordered system

**Definition (RCA*₀)**

An ordered system \((M, e, <)\) is said to be inductive if it satisfies the following:

- (induction): for any \(Z \subseteq M\),

\[ (e \in Z \land \forall x \in Z (x' \in Z)) \rightarrow \forall x (x \in Z). \]

An ordered system \((M, e, <)\) is said to be strongly inductive if it satisfies the following:

- (maximal/minimal element): for any \(Z \subseteq M\), if there exists \(a \in M\) such that \(\forall x \in Z \; x < a\), then, \(\max Z\) and \(\min Z\) exist.

**Proposition (RCA*₀)**

Strongly inductive ordered system is inductive.
**Definition (RCA\(_0^*\))**

An ordered system \((M, e, <)\) is said to be inductive if it satisfies the following:

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**Proposition (RCA\(_0^*\))**

*Strongly inductive ordered system is inductive.*
An ordered system \((M, e, <)\) is said to be isomorphic to \(\mathbb{N}\) if there exists a bijective homomorphism \(\Phi : M \rightarrow \mathbb{N}\).

An ordered system \((M, e, <)\) is said to be almost isomorphic to \(\mathbb{N}\) if for any \(x \in M\) there exists a sequence \(\langle a_i \mid i \leq k \rangle\) such that \(a_0 = e, a_k = x\) and \(a_{i+1} = a_i'\) for any \(i < k\).
Inductive ordered system

Definition (RCA₀*)

An ordered system \((M, e, \prec)\) is said to be isomorphic to \(\mathbb{N}\) if there exists a bijective homomorphism \(\Phi : M \rightarrow \mathbb{N}\).

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Then,
Isomorphism of ordered systems

Theorem

The following are equivalent over RCA\(^*\).

1. ACA\(_0\).
2. Every inductive ordered system is isomorphic to \(\mathbb{N}\).
3. Every strongly inductive ordered system is isomorphic to \(\mathbb{N}\).
4. Every inductive ordered system which is almost isomorphic to \(\mathbb{N}\) is isomorphic to \(\mathbb{N}\).
5. Every strongly inductive ordered system which is almost isomorphic to \(\mathbb{N}\) is isomorphic to \(\mathbb{N}\).

Proof.

1 \(\rightarrow\) 2 is easy. 2 \(\rightarrow\) 3, 4, 5 is trivial. 5 \(\rightarrow\) 1...
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The following are equivalent over RCA$_0^*$.  

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5. Every strongly inductive ordered system which is almost isomorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$.

Proof.

$1 \rightarrow 2$ is easy. $2 \rightarrow 3,4,5$ is trivial. $5 \rightarrow 1$...
Definition (RCA$^*_0$)

Let $(L, <_L)$ be a countable linear ordering.

1. $(L, <_L)$ is said to be *pseudofinite* if every nonempty subset of $L$ has a first element and a last element.

2. Let $X \subseteq L$. A *left (right) boundary point* of $X$ is an element $a \in X$ such that there is no $c <_L a$ ($c >_L a$) with

$$\{b \mid c \leq_L b \leq_L a\} \subseteq X \quad (\{b \mid a \leq_L b \leq_L c\} \subseteq X).$$

$L, <_L$ is said to be *quasifinite* if each nonempty subset of $L$ has a left boundary point and a right boundary point.
PFO and QFO

Definition

1. **PFO**: every countable pseudofinite linear ordering is finite.
   \[ PFO_0 = RCA_0 + PFO, \quad PFO^*_0 = RCA^*_0 + PFO. \]

2. **QFO**: every countable pseudofinite linear ordering is finite.
   \[ QFO_0 = RCA_0 + QFO, \quad QFO^*_0 = RCA^*_0 + QFO. \]

Proposition (RCA\(^*_0\), by Shore and Hirschfeldt)

**PFO**\(_0\) is equivalent to **ADS**\(_0\) = RCA\(_0\) + **ADS**.

Proposition (RCA\(^*_0\))

**QFO**\(^*_0\) implies **PFO**\(^*_0\) and **WKL**\(^*_0\).
**QFO**\(_0\) implies **ACA**\(_0\) [R. Shore, in private communication].
PFO and QFO

Definition

1. PFO: every countable pseudofinite linear ordering is finite.
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\[ PFO_0 \text{ is equivalent to } ADS_0 = RCA_0 + ADS. \]

Proposition (RCA\(^*_0\))

\[ QFO_0^* \text{ implies } PFO_0^* \text{ and } WKL_0^*. \]
\[ QFO_0 \text{ implies } ACA_0 \text{ [R. Shore, in private communication].} \]


**PFO and QFO**

**Definition**

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   \[\text{PFO}_0 = \text{RCA}_0 + \text{PFO}, \quad \text{PFO}_0^* = \text{RCA}_0^* + \text{PFO}.\]

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**Proposition (RCA\(_0^*\))**

\[\text{QFO}_0^* \text{ implies } \text{PFO}_0^* \text{ and } \text{WKL}_0^*.\]

\[\text{QFO}_0 \text{ implies } \text{ACA}_0 \quad [\text{R. Shore, in private communication}].\]
Almost isomorphism of ordered systems

Theorem

The following are equivalent over $\text{RCA}_0^*$.

1. $\text{PFO}_0^*$.
2. Every strongly inductive ordered system is almost isomorphic to $\mathbb{N}$.

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The following are equivalent over $\text{RCA}_0^*$.

1. $\text{QFO}_0^*$.
2. Every inductive ordered system is almost isomorphic to $\mathbb{N}$. 
Finally, we characterize the natural number system by using an order and a successor function.

**Definition (RCA$_0^*$)**

An ordered successor system is a quadruple $(M, e, f, <)$ where $(M, e, <)$ is an ordered system, and $f$ is its successor function.

**Definition (RCA$_0^*$)**

- An ordered successor system $(M, e, f, <)$ is said to be inductive if $(M, e, <)$ is inductive.
- An ordered successor system $(M, e, f, <)$ is said to be strongly inductive if $(M, e, <)$ is strongly inductive.
Inductive ordered successor system

Finally, we characterize the natural number system by using an order and a successor function.

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Finally, we characterize the natural number system by using an order and a successor function.

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An ordered successor system is a quadruple \((M, e, f, <)\) where \((M, e, <)\) is an ordered system, and \(f\) is its successor function.

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Inductive ordered successor system

Theorem

The following are equivalent over RCA$^*_0$.

1. WKL$^*_0$.
2. Every inductive ordered successor system is almost isomorphic to $\mathbb{N}$.

Theorem

The following are equivalent over RCA$^*_0$.

1. RCA$_0$.
2. Every strongly inductive ordered successor system is isomorphic to $\mathbb{N}$.
3. Every (strongly) inductive ordered successor system which is almost isomorphic to $\mathbb{N}$ is isomorphic to $\mathbb{N}$. 
### Inductive ordered successor system

**Theorem**

The following are equivalent over \( \text{RCA}_0^* \).

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### Summary

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**Table:** Summary of results.
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**Table:** Summary of results.
### Open questions

**PFO**\(_0\) is equivalent to RCA\(_0\) + ADS.
Actually it is equivalent to RCA\(_\ast\)\(_0\) + ADS since ADS implies \(\Sigma^0_1\)-IND.

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**QFO**\(_0\) is equivalent to ACA\(_0\).
QFO\(_\ast\)\(_0\) implies both of ADS\(_\ast\)\(_0\) and WKL\(_\ast\)\(_0\).
However, we don’t know whether QFO\(_\ast\)\(_0\) proves \(\Sigma^0_1\)-IND or not.

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What is the proof-theoretic strength of PFO\textsubscript{0} + ADS? Is it $\Pi^1_1$ conservative over RCA\textsubscript{0}? 

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Question

What is the proof-theoretic strength of QFO\textsubscript{0}?
PFO₀ is equivalent to RCA₀ + ADS. Actually it is equivalent to RCA₀⁺ + ADS since ADS implies Σ₁⁰-IND.

**Question**

What is the proof-theoretic strength of PFO₀⁺? Is it Π₁¹ conservative over RCA₀⁺?

QFO₀ is equivalent to ACA₀. QFO₀⁺ implies both of ADS₀⁺ and WKL₀⁺. However, we don’t know whether QFO₀⁺ proves Σ₁⁰-IND or not.

**Question**

What is the proof-theoretic strength of QFO₀⁺?
PFO₀ is equivalent to RCA₀ + ADS. Actually it is equivalent to RCA₀⁺ + ADS since ADS implies $\Sigma^0_1$-IND.

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What is the proof-theoretic strength of PFO₀⁺? Is it $\Pi^1_1$ conservative over RCA₀⁺?

QFO₀ is equivalent to ACA₀. QFO₀⁺ implies both of ADS₀⁺ and WKL₀⁺. However, we don’t know whether QFO₀⁺ proves $\Sigma^0_1$-IND or not.

Question

What is the proof-theoretic strength of QFO₀⁺?
Open questions

PFO_0 is equivalent to RCA_0 + ADS. Actually it is equivalent to RCA_0^* + ADS since ADS implies $\Sigma^0_1$-IND.

Question

What is the proof-theoretic strength of PFO_0^*? Is it $\Pi^1_1$ conservative over RCA_0^*?

QFO_0 is equivalent to ACA_0.
QFO_0^* implies both of ADS_0^* and WKL_0^*.
However, we don’t know whether QFO_0^* proves $\Sigma^0_1$-IND or not.

Question

What is the proof-theoretic strength of QFO_0^*?
Open questions

Almost isomorphism for strongly inductive ordered successor system is provable from $\text{RCA}_0$, $\text{WKL}_0^*$ and $\text{PFO}_0^*$.

**Question**

What is the reverse-mathematical status of the statement that every strongly inductive ordered successor system is almost isomorphic to $\mathbb{N}$? Is it provable within $\text{RCA}_0^*$?

**Question**

Is the natural number system $\mathbb{N}$ second-order characterizable within $\text{RCA}_0^*$?

For this, we want a second-order statement $\varphi$ such that

- $\text{RCA}_0^*$ proves $\mathbb{N}$ satisfies $\varphi$.
- $\text{RCA}_0^*$ proves the categoricity theorem for $\varphi$. 
Open questions

Almost isomorphism for strongly inductive ordered successor system is provable from $\text{RCA}_0$, $\text{WKL}_0^*$ and $\text{PFO}_0^*$.

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- $\text{RCA}_0^*$ proves $\mathbb{N}$ satisfies $\varphi$.
- $\text{RCA}_0^*$ proves the categoricity theorem for $\varphi$. 
Open questions

Almost isomorphism for strongly inductive ordered successor system is provable from $\text{RCA}_0$, $\text{WKL}_0^*$ and $\text{PFO}_0^*$.

**Question**

What is the reverse-mathematical status of the statement that every strongly inductive ordered successor system is almost isomorphic to $\mathbb{N}$? Is it provable within $\text{RCA}_0^*$?

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